

# Higher-dimensional absolute versions of symmetric, Frobenius, and quasi-Frobenius algebras

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## Abstract

In this paper, we define and discuss higher-dimensional and absolute versions of symmetric, Frobenius, and quasi-Frobenius algebras. In particular, we compare these with the relative notions defined by Scheja and Storch. We also prove the validity of codimension two-argument for modules over a coherent sheaf of algebras with a 2-canonical module, generalizing a result of the author.

## 1. Introduction

**(1.1)** Let  $(R, \mathfrak{m})$  be a semilocal Noetherian commutative ring, and  $\Lambda$  a module-finite  $R$ -algebra. In [6], we defined the canonical module  $K_\Lambda$  of  $\Lambda$ . The purpose of this paper is two fold, each of which is deeply related to  $K_\Lambda$ .

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(1.2) In the first part, we define and discuss higher-dimensional and absolute notions of symmetric, Frobenius, and quasi-Frobenius algebras and their non-Cohen–Macaulay versions. In commutative algebra, the non-Cohen–Macaulay version of Gorenstein ring is known as quasi-Gorenstein rings. What we discuss here is a non-commutative version of such rings. Scheja and Storch [7] discussed a relative notion, and our definition is absolute in the sense that it depends only on  $\Lambda$  and is independent of the choice of  $R$ . If  $R$  is local, our quasi-Frobenius property agrees with Gorensteinness discussed by Goto and Nishida [1], see Proposition 3.7 and Corollary 3.8.

(1.3) In the second part, we show that the codimension-two argument using the existence of 2-canonical modules in [4] is still valid in non-commutative settings. For the definition of an  $n$ -canonical module, see (2.8). Codimension-two argument, which states (roughly speaking) that removing a closed subset of codimension two or more does not change the category of coherent sheaves which satisfy Serre’s  $(S'_2)$  condition, is sometimes used in algebraic geometry, commutative algebra and invariant theory. For example, information on the canonical sheaf and the class group is retained when we remove the singular locus of a normal variety over an algebraically closed field, and then these objects are respectively grasped as the top exterior power of the cotangent bundle and the Picard group of a smooth variety. In [4], almost principal bundles are studied. They are principal bundles after removing closed subsets of codimension two or more.

We prove the following. Let  $X$  be a locally Noetherian scheme,  $U$  an open subset of  $X$  such that  $\text{codim}_X(X \setminus U) \geq 2$ . Let  $i : U \rightarrow X$  be the inclusion. Let  $\Lambda$  be a coherent  $\mathcal{O}_X$ -algebra. If  $X$  possesses a 2-canonical module  $\omega$ , then the inverse image  $i^*$  induces the equivalence between the category of coherent right  $\Lambda$ -modules which satisfy the  $(S'_2)$  condition and the category of coherent right  $i^*\Lambda$ -modules which satisfy the  $(S'_2)$  condition. The quasi-inverse is given by the direct image  $i_*$ . What was proved in [4] was the case that  $\Lambda = \mathcal{O}_X$ . If, moreover,  $\omega = \mathcal{O}_X$  (that is to say,  $X$  satisfy the  $(S_2)$  and  $(G_1)$  condition), then the assertion has been well-known, see [3].

(1.4) 2-canonical modules are ubiquitous in algebraic geometry. If  $\mathbb{I}$  is a dualizing complex of a Noetherian scheme  $X$ , then the lowest non-vanishing cohomology group of  $\mathbb{I}$  is semicanonical. A rank-one reflexive sheaf over a normal variety is 2-canonical.

(1.5) Section 2 is for preliminaries. Section 3 is devoted to the discussion of the first theme mentioned in the paragraph (1.2), while Section 4 is for the second theme mentioned in (1.3).

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The essential part of this paper has first appeared as [5, sections 9–10]. When it is published as [6], they have been removed after the requirement to shorten the paper (also, the title has been changed slightly). Here we revive them as an independent paper.

## 2. Preliminaries

(2.1) Throughout this paper,  $R$  denotes a Noetherian commutative ring. For a module-finite  $R$ -algebra  $\Lambda$ , a  $\Lambda$ -module means a left  $\Lambda$ -module.  $\Lambda^{\text{op}}$  denotes the opposite algebra of  $\Lambda$ , and thus a  $\Lambda^{\text{op}}$ -module is identified with a right  $\Lambda$ -module. A  $\Lambda$ -bimodule means a  $\Lambda \otimes_R \Lambda^{\text{op}}$ -module. The category of finite  $\Lambda$ -modules is denoted by  $\Lambda \text{ mod}$ . The category  $\Lambda^{\text{op}} \text{ mod}$  is also denoted by  $\text{mod } \Lambda$ .

(2.2) Let  $(R, \mathfrak{m})$  be semilocal and  $\Lambda$  be a module-finite  $R$ -algebra. For an  $R$ -module  $M$ , the  $\mathfrak{m}$ -adic completion of  $M$  is denoted by  $\hat{M}$ . For a finite  $\Lambda$ -module  $M$ , by  $\dim M$  or  $\dim_{\Lambda} M$  we mean  $\dim_R M$ , which is independent of the choice of  $R$ . By  $\text{depth } M$  or  $\text{depth}_{\Lambda} M$  we mean  $\text{depth}_R(\mathfrak{m}, M)$ , which is independent of  $R$ . We say that  $M$  is globally Cohen–Macaulay (GCM for short) if  $\dim M = \text{depth } M$ . We say that  $M$  is globally maximal Cohen–Macaulay (GMCM for short) if  $\dim \Lambda = \text{depth } M$ . If  $R$  happens to be local, then  $M$  is GCM (resp. GMCM) if and only if  $M$  is Cohen–Macaulay (resp. maximal Cohen–Macaulay) as an  $R$ -module.

(2.3) For  $M \in \Lambda \text{ mod}$ , we say that  $M$  satisfies  $(S'_n)^{\Lambda, R}$ ,  $(S'_n)^R$  or  $(S'_n)$  if  $\text{depth}_{R_P} M_P \geq \min(n, \text{ht}_R P)$  for every  $P \in \text{Spec } R$  (this notion depends on  $R$ ).

(2.4) Let  $X$  be a locally Noetherian scheme and  $\Lambda$  a coherent  $\mathcal{O}_X$ -algebra. For a coherent  $\Lambda$ -module  $\mathcal{M}$ , we say that  $\mathcal{M}$  satisfies  $(S'_n)$  or  $(S'_n)^{\Lambda, X}$ , or sometimes  $\mathcal{M} \in (S'_n)^{\Lambda, X}$ , if  $\text{depth}_{\mathcal{O}_{X,x}} \mathcal{M}_x \geq \min(n, \dim \mathcal{O}_{X,x})$  for every  $x \in X$ .

(2.5) Assume that  $(R, \mathfrak{m})$  is complete semilocal, and  $\Lambda \neq 0$  a module-finite  $R$ -algebra. Let  $\mathbb{I}$  be a normalized dualizing complex of  $R$ . The lowest non-vanishing cohomology group  $\mathrm{Ext}_R^{-s}(\Lambda, \mathbb{I})$  ( $\mathrm{Ext}_R^i(\Lambda, \mathbb{I}) = 0$  for  $i < -s$ ) is denoted by  $K_\Lambda$ , and is called the *canonical module* of  $\Lambda$ . If  $\Lambda = 0$ , then we define that  $K_\Lambda = 0$ . For basics on the canonical modules, we refer the reader to [6]. Note that  $K_\Lambda$  depends only on  $\Lambda$ , and is independent of  $R$ .

(2.6) Assume that  $(R, \mathfrak{m})$  is semilocal which may not be complete. We say that a finitely generated  $\Lambda$ -bimodule  $K$  is a *canonical module* of  $\Lambda$  if  $\hat{K}$  is isomorphic to the canonical module  $K_{\hat{\Lambda}}$  as a  $\hat{\Lambda}$ -bimodule. It is unique up to isomorphisms, and denoted by  $K_\Lambda$ . We say that  $K \in \mathrm{mod} \Lambda$  is a right (resp. left) canonical module of  $\Lambda$  if  $\hat{K}$  is isomorphic to  $K_{\hat{\Lambda}}$  in  $\mathrm{mod} \hat{\Lambda}$  (resp.  $\hat{\Lambda} \mathrm{mod}$ ). If  $K_\Lambda$  exists, then  $K$  is a right canonical module if and only if  $K \cong K_\Lambda$  in  $\mathrm{mod} \Lambda$ .

(2.7) We say that  $\omega$  is an  $R$ -semicanonical right  $\Lambda$ -module if for any  $P \in \mathrm{Spec} R$ ,  $R_P \otimes_R \omega$  is the right canonical module  $R_P \otimes_R \Lambda$  for any  $P \in \mathrm{supp}_R \omega$ .

(2.8) Let  $C \in \mathrm{mod} \Lambda$ . We say that  $C$  is an  $n$ -canonical right  $\Lambda$ -module over  $R$  if  $C \in (S'_n)^R$ , and for each  $P \in \mathrm{Spec} R$  with  $\mathrm{ht} P < n$ , we have that  $C_P$  is an  $R_P$ -semicanonical right  $\Lambda_P$ -module.

### 3. Symmetric and Frobenius algebras

(3.1) Let  $(R, \mathfrak{m})$  be a Noetherian semilocal ring, and  $\Lambda$  a module-finite  $R$ -algebra. Let  $K_\Lambda$  denote the canonical module of  $\Lambda$ , see [6].

We say that  $\Lambda$  is *quasi-symmetric* if  $\Lambda$  is the canonical module of  $\Lambda$ . That is,  $\Lambda \cong K_\Lambda$  as  $\Lambda$ -bimodules. It is called *symmetric* if it is quasi-symmetric and GCM. Note that  $\Lambda$  is quasi-symmetric (resp. symmetric) if and only if  $\hat{\Lambda}$  is so. Note also that quasi-symmetric and symmetric are absolute notion, and is independent of the choice of  $R$  in the sense that the definition does not change when we replace  $R$  by the center of  $\Lambda$ , because  $K_\Lambda$  is independent of the choice of  $R$ .

(3.2) For (non-semilocal) Noetherian ring  $R$ , we say that  $\Lambda$  is locally quasi-symmetric (resp. locally symmetric) over  $R$  if for any  $P \in \mathrm{Spec} R$ ,  $\Lambda_P$  is a quasi-symmetric (resp. symmetric)  $R_P$ -algebra. This is equivalent to say that for any maximal ideal  $\mathfrak{m}$  of  $R$ ,  $\Lambda_{\mathfrak{m}}$  is quasi-symmetric (resp. symmetric), see [6, (7.6)].

In the case that  $(R, \mathfrak{m})$  is semilocal,  $\Lambda$  is locally quasi-symmetric (resp. locally symmetric) over  $R$  if it is quasi-symmetric (resp. symmetric), but the converse is not true in general.

**Lemma 3.3.** *Let  $(R, \mathfrak{m})$  be a Noetherian semilocal ring, and  $\Lambda$  a module-finite  $R$ -algebra. Then the following are equivalent.*

**1**  $\Lambda_\Lambda$  is the right canonical module of  $\Lambda$ .

**2**  ${}_\Lambda \Lambda$  is the left canonical module of  $\Lambda$ .

*Proof.* We may assume that  $R$  is complete. Then replacing  $R$  by a Noether normalization of  $R/\text{ann}_R \Lambda$ , we may assume that  $R$  is regular and  $\Lambda$  is a faithful  $R$ -module.

We prove **1** $\Rightarrow$ **2**. By [6, Lemma 5.10],  $K_\Lambda$  satisfies  $(S'_2)^R$ . By assumption,  $\Lambda_\Lambda$  satisfies  $(S'_2)^R$ . As  $R$  is regular and  $\dim R = \dim \Lambda$ ,  $K_\Lambda = \Lambda^* = \text{Hom}_R(\Lambda, R)$ . So we get an  $R$ -linear map

$$\varphi : \Lambda \otimes_R \Lambda \rightarrow R$$

such that  $\varphi(ab \otimes c) = \varphi(a \otimes bc)$  and that the induced map  $h : \Lambda \rightarrow \Lambda^*$  given by  $h(a)(c) = \varphi(a \otimes c)$  is an isomorphism (in  $\text{mod } \Lambda$ ). Now  $\varphi$  induces a homomorphism  $h' : \Lambda \rightarrow \Lambda^*$  in  $\Lambda \text{ mod}$  given by  $h'(c)(a) = \varphi(a \otimes c)$ . To verify that this is an isomorphism, as  $\Lambda$  and  $\Lambda^*$  are reflexive  $R$ -modules, we may localize at a prime  $P$  of  $R$  of height at most one, and then take a completion, and hence we may further assume that  $\dim R \leq 1$ . Then  $\Lambda$  is a finite free  $R$ -module, and the matrices of  $h$  and  $h'$  are transpose each other. As the matrix of  $h$  is invertible, so is that of  $h'$ , and  $h'$  is an isomorphism.

**2** $\Rightarrow$ **1** follows from **1** $\Rightarrow$ **2**, considering the opposite ring.  $\square$

**Definition 3.4.** Let  $(R, \mathfrak{m})$  be semilocal. We say that  $\Lambda$  is a *pseudo-Frobenius  $R$ -algebra* if the equivalent conditions of Lemma 3.3 are satisfied. If  $\Lambda$  is GCM in addition, then it is called a *Frobenius  $R$ -algebra*. Note that these definitions are independent of the choice of  $R$ . Moreover,  $\Lambda$  is pseudo-Frobenius (resp. Frobenius) if and only if  $\hat{\Lambda}$  is so. For a general  $R$ , we say that  $\Lambda$  is locally pseudo-Frobenius (resp. locally Frobenius) over  $R$  if  $\Lambda_P$  is pseudo-Frobenius (resp. Frobenius) for  $P \in \text{Spec } R$ .

**Lemma 3.5.** *Let  $(R, \mathfrak{m})$  be semilocal. Then the following are equivalent.*

**1**  $(K_{\hat{\Lambda}})_{\hat{\Lambda}}$  is projective in  $\text{mod } \hat{\Lambda}$ .

**2**  $\hat{\Lambda}(K_{\hat{\Lambda}})$  is projective in  $\hat{\Lambda} \text{ mod}$ ,

where  $\hat{\phantom{x}}$  denotes the  $\mathfrak{m}$ -adic completion.

*Proof.* We may assume that  $(R, \mathfrak{m}, k)$  is complete regular local and  $\Lambda$  is a faithful  $R$ -module. Let  $\bar{\phantom{x}}$  denote the functor  $k \otimes_R \phantom{x}$ . Then  $\bar{\Lambda}$  is a finite dimensional  $k$ -algebra. So  $\text{mod } \bar{\Lambda}$  and  $\bar{\Lambda} \text{ mod}$  have the same number of simple modules, say  $n$ . An indecomposable projective module in  $\text{mod } \Lambda$  is nothing but the projective cover of a simple module in  $\text{mod } \bar{\Lambda}$ . So  $\text{mod } \Lambda$  and  $\Lambda \text{ mod}$  have  $n$  indecomposable projectives. Now  $\text{Hom}_R(\phantom{x}, R)$  is an equivalence between  $\text{add}(K_{\Lambda})_{\Lambda}$  and  $\text{add } {}_{\Lambda}\Lambda$ . It is also an equivalence between  $\text{add } {}_{\Lambda}(K_{\Lambda})$  and  $\text{add } \Lambda_{\Lambda}$ . So both  $\text{add}(K_{\Lambda})_{\Lambda}$  and  $\text{add } {}_{\Lambda}(K_{\Lambda})$  also have  $n$  indecomposables. So **1** is equivalent to  $\text{add}(K_{\Lambda})_{\Lambda} = \text{add } \Lambda_{\Lambda}$ . **2** is equivalent to  $\text{add } {}_{\Lambda}(K_{\Lambda}) = \text{add } \Lambda_{\Lambda}$ . So **1**  $\Leftrightarrow$  **2** is proved simply applying the duality  $\text{Hom}_R(\phantom{x}, R)$ .  $\square$

**(3.6)** Let  $(R, \mathfrak{m})$  be semilocal. If the equivalent conditions in Lemma 3.5 are satisfied, then we say that  $\Lambda$  is *pseudo-quasi-Frobenius*. If it is GCM in addition, then we say that it is *quasi-Frobenius*. These definitions are independent of the choice of  $R$ . Note that  $\Lambda$  is pseudo-quasi-Frobenius (resp. quasi-Frobenius) if and only if  $\hat{\Lambda}$  is so.

**Proposition 3.7.** *Let  $(R, \mathfrak{m})$  be semilocal. Then the following are equivalent.*

- 1**  $\Lambda$  is quasi-Frobenius.
- 2**  $\Lambda$  is GCM, and  $\dim \Lambda = \text{idim } {}_{\Lambda}\Lambda$ , where  $\text{idim}$  denotes the injective dimension.
- 3**  $\Lambda$  is GCM, and  $\dim \Lambda = \text{idim } \Lambda_{\Lambda}$ .

*Proof.* **1**  $\Rightarrow$  **2**. By definition,  $\Lambda$  is GCM. To prove that  $\dim \Lambda = \text{idim } {}_{\Lambda}\Lambda$ , we may assume that  $R$  is local. Then by [1, (3.5)], we may assume that  $R$  is complete. Replacing  $R$  by the Noetherian normalization of  $R/\text{ann}_R \Lambda$ , we may assume that  $R$  is a complete regular local ring of dimension  $d$ , and  $\Lambda$  its maximal Cohen–Macaulay (that is, finite free) module. As  $\text{add } {}_{\Lambda}\Lambda = \text{add } {}_{\Lambda}(K_{\Lambda})$  by the proof of Lemma 3.5, it suffices to prove  $\text{idim } {}_{\Lambda}(K_{\Lambda}) = d$ . Let  $\mathbb{I}_R$  be the minimal injective resolution of the  $R$ -module  $R$ . Then  $\mathbb{J} = \text{Hom}_R(\Lambda, \mathbb{I}_R)$  is an injective resolution of  $K_{\Lambda} = \text{Hom}_R(\Lambda, R)$  as a left  $\Lambda$ -module. As the length of  $\mathbb{J}$  is  $d$  and

$$\text{Ext}_{\Lambda}^d(\Lambda/\mathfrak{m}\Lambda, K_{\Lambda}) \cong \text{Ext}_R^d(\Lambda/\mathfrak{m}\Lambda, R) \neq 0,$$

we have that  $\text{idim}_\Lambda(K_\Lambda) = d$ .

**2 $\Rightarrow$ 1.** We may assume that  $R$  is complete regular local and  $\Lambda$  is maximal Cohen–Macaulay. By [1, (3.6)], we may further assume that  $R$  is a field. Then  ${}_\Lambda\Lambda$  is injective. So  $(K_\Lambda)_\Lambda = \text{Hom}_R(\Lambda, R)$  is projective, and  $\Lambda$  is quasi-Frobenius, see [8, (IV.3.7)].

**1 $\Leftrightarrow$ 3** is proved similarly.  $\square$

**Corollary 3.8.** *Let  $R$  be arbitrary. Then the following are equivalent.*

- 1 *For any  $P \in \text{Spec } R$ ,  $\Lambda_P$  is quasi-Frobenius.*
- 2 *For any maximal ideal  $\mathfrak{m}$  of  $R$ ,  $\Lambda_{\mathfrak{m}}$  is quasi-Frobenius.*
- 3  *$\Lambda$  is a Gorenstein  $R$ -algebra in the sense that  $\Lambda$  is a Cohen–Macaulay  $R$ -module, and  $\text{idim}_{\Lambda_P} \Lambda_P = \dim \Lambda_P$  for any  $P \in \text{Spec } R$ .*

*Proof.* **1 $\Rightarrow$ 2** is trivial.

**2 $\Rightarrow$ 3.** By Proposition 3.7, we have  $\text{idim}_{\Lambda_{\mathfrak{m}}} \Lambda_{\mathfrak{m}} = \dim \Lambda_{\mathfrak{m}}$  for each  $\mathfrak{m}$ . Then by [1, (4.7)],  $\Lambda$  is a Gorenstein  $R$ -algebra.

**3 $\Rightarrow$ 1** follows from Proposition 3.7.  $\square$

**(3.9)** Let  $R$  be arbitrary. We say that  $\Lambda$  is a *quasi-Gorenstein  $R$ -algebra* if  $\Lambda_P$  is pseudo-quasi-Frobenius for each  $P \in \text{Spec } R$ .

**Definition 3.10** (Scheja–Storch [7]). Let  $R$  be general. We say that  $\Lambda$  is symmetric (resp. Frobenius) relative to  $R$  if  $\Lambda$  is  $R$ -projective, and  $\Lambda^* := \text{Hom}_R(\Lambda, R)$  is isomorphic to  $\Lambda$  as a  $\Lambda$ -bimodule (resp. as a right  $\Lambda$ -module). It is called quasi-Frobenius relative to  $R$  if the right  $\Lambda$ -module  $\Lambda^*$  is projective.

**Lemma 3.11.** *Let  $(R, \mathfrak{m})$  be local.*

- 1 *If  $\dim \Lambda = \dim R$ ,  $R$  is quasi-Gorenstein, and  $\Lambda^* \cong \Lambda$  as  $\Lambda$ -bimodules (resp.  $\Lambda^* \cong \Lambda$  as right  $\Lambda$ -modules,  $\Lambda^*$  is projective as a right  $\Lambda$ -module), then  $\Lambda$  is quasi-symmetric (resp. pseudo-Frobenius, pseudo-quasi-Frobenius).*
- 2 *If  $R$  is Gorenstein and  $\Lambda$  is symmetric (resp. Frobenius, quasi-Frobenius) relative to  $R$ , then  $\Lambda$  is symmetric (resp. Frobenius, quasi-Frobenius).*
- 3 *If  $\Lambda$  is nonzero and  $R$ -projective, then  $\Lambda$  is quasi-symmetric (resp. pseudo-Frobenius, pseudo-quasi-Frobenius) if and only if  $R$  is quasi-Gorenstein and  $\Lambda$  is symmetric (resp. Frobenius, quasi-Frobenius) relative to  $R$ .*

- 4** *If  $\Lambda$  is nonzero and  $R$ -projective, then  $\Lambda$  is symmetric (resp. Frobenius, quasi-Frobenius) if and only if  $R$  is Gorenstein and  $\Lambda$  is symmetric (resp. Frobenius, quasi-Frobenius) relative to  $R$ .*

*Proof.* We can take the completion, and we may assume that  $R$  is complete local.

- 1.** Let  $d = \dim \Lambda = \dim R$ , and let  $\mathbb{I}$  be the normalized dualizing complex (see [6, (5.2)]) of  $R$ . Then

$$K_\Lambda = \operatorname{Ext}_R^{-d}(\Lambda, \mathbb{I}) \cong \operatorname{Hom}_R(\Lambda, H^{-d}(\mathbb{I})) \cong \operatorname{Hom}(\Lambda, K_R) \cong \operatorname{Hom}(\Lambda, R) = \Lambda^*$$

as  $\Lambda$ -bimodules, and the result follows.

- 2.** We may assume that  $\Lambda$  is nonzero. As  $R$  is Cohen–Macaulay and  $\Lambda$  is a finite projective  $R$ -module,  $\Lambda$  is a maximal Cohen–Macaulay  $R$ -module. By **1**, the result follows.

- 3.** The ‘if’ part follows from **1**. We prove the ‘only if’ part. As  $\Lambda$  is  $R$ -projective and nonzero,  $\dim \Lambda = \dim R$ . As  $\Lambda$  is  $R$ -finite free,  $K_\Lambda \cong \operatorname{Hom}_R(\Lambda, K_R) \cong \Lambda^* \otimes_R K_R$ . As  $K_\Lambda$  is  $R$ -free and  $\Lambda^* \otimes_R K_R$  is nonzero and is isomorphic to a direct sum of copies of  $K_R$ , we have that  $K_R$  is  $R$ -projective, and hence  $R$  is quasi-Gorenstein, and  $K_R \cong R$ . Hence  $K_\Lambda \cong \Lambda^*$ , and the result follows.

**4** follows from **3** easily. □

**(3.12)** Let  $(R, \mathfrak{m})$  be semilocal. Let a finite group  $G$  act on  $\Lambda$  by  $R$ -algebra automorphisms. Let  $\Omega = \Lambda * G$ , the twisted group algebra. That is,  $\Omega = \Lambda \otimes_R RG = \bigoplus_{g \in G} \Lambda g$  as an  $R$ -module, and the product of  $\Omega$  is given by  $(ag)(a'g') = (a(ga'))(gg')$  for  $a, a' \in \Lambda$  and  $g, g' \in G$ . This makes  $\Omega$  a module-finite  $R$ -algebra.

**(3.13)** We simply call an  $RG$ -module a  $G$ -module. We say that  $M$  is a  $(G, \Lambda)$ -module if  $M$  is a  $G$ -module,  $\Lambda$ -module, the  $R$ -module structures coming from that of the  $G$ -module structure and the  $\Lambda$ -module structure agree, and  $g(am) = (ga)(gm)$  for  $g \in G$ ,  $a \in \Lambda$ , and  $m \in M$ . A  $(G, \Lambda)$ -module and an  $\Omega$ -module are one and the same thing.

**(3.14)** By the action  $((a \otimes a')g)a_1 = a(ga_1)a'$ , we have that  $\Lambda$  is a  $(\Lambda \otimes \Lambda^{\operatorname{op}}) * G$ -module in a natural way. So it is an  $\Omega$ -module by the action  $(ag)a_1 = a(ga_1)$ . It is also a right  $\Omega$ -module by the action  $a_1(ag) = g^{-1}(a_1a)$ . If the action of  $G$  on  $\Lambda$  is trivial, then these actions make an  $\Omega$ -bimodule.



**(3.15)** Given an  $\Omega$ -module  $M$  and an  $RG$ -module  $V$ ,  $M \otimes_R V$  is an  $\Omega$ -module by  $(ag)(m \otimes v) = (ag)m \otimes gv$ .  $\text{Hom}_R(M, V)$  is a right  $\Omega$ -module by  $(\varphi(ag))(m) = g^{-1}\varphi(agm)$ . It is easy to see that the standard isomorphism

$$\text{Hom}_R(M \otimes_R V, W) \rightarrow \text{Hom}_R(M, \text{Hom}_R(V, W))$$

is an isomorphism of right  $\Omega$ -modules for a left  $\Omega$ -module  $M$  and  $G$ -modules  $V$  and  $W$ .

**(3.16)** Now consider the case  $\Lambda = R$ . Then the pairing  $\phi : RG \otimes_R RG \rightarrow R$  given by  $\phi(g \otimes g') = \delta_{gg', e}$  (Kronecker's delta) is non-degenerate, and induces an  $RG$ -bimodule isomorphism  $\Omega = RG \rightarrow (RG)^* = \Omega^*$ . As  $\Omega = RG$  is a finite free  $R$ -module, we have that  $\Omega = RG$  is symmetric relative to  $R$ .

**Lemma 3.17.** *If  $\Lambda$  is quasi-symmetric (resp. symmetric) and the action of  $G$  on  $\Lambda$  is trivial, then  $\Omega$  is quasi-symmetric (resp. symmetric).*

*Proof.* Taking the completion, we may assume that  $R$  is complete. Then replacing  $R$  by a Noether normalization of  $R/\text{ann}_R \Lambda$ , we may assume that  $R$  is a regular local ring, and  $\Lambda$  is a faithful  $R$ -module. As the action of  $G$  on  $\Lambda$  is trivial,  $\Omega = \Lambda \otimes_R RG$  is quasi-symmetric (resp. symmetric), as can be seen easily.  $\square$

**(3.18)** In particular, if  $\Lambda$  is commutative quasi-Gorenstein (resp. Gorenstein) and the action of  $G$  on  $\Lambda$  is trivial, then  $\Omega = \Lambda G$  is quasi-symmetric (resp. symmetric).

**(3.19)** In general,  ${}_{\Omega}\Omega \cong \Lambda \otimes_R RG$  as  $\Omega$ -modules.

**Lemma 3.20.** *Let  $M$  and  $N$  be right  $\Omega$ -modules, and let  $\varphi : M \rightarrow N$  be a homomorphism of right  $\Lambda$ -modules. Then  $\psi : M \otimes RG \rightarrow N \otimes RG$  given by  $\psi(m \otimes g) = g(\varphi(g^{-1}m)) \otimes g$  is an  $\Omega$ -homomorphism. In particular,*

- 1 *If  $\varphi$  is a  $\Lambda$ -isomorphism, then  $\psi$  is an  $\Omega$ -isomorphism.*
- 2 *If  $\varphi$  is a split monomorphism in  $\text{mod } \Lambda$ , then  $\psi$  is a split monomorphism in  $\text{mod } \Omega$ .*

*Proof.* Straightforward.  $\square$

**Proposition 3.21.** *Let  $G$  be a finite group acting on  $\Lambda$ . Set  $\Omega := \Lambda * G$ .*

- 1 *If the action of  $G$  on  $\Lambda$  is trivial and  $\Lambda$  is quasi-symmetric (resp. symmetric), then so is  $\Omega$ .*

**2** If  $\Lambda$  is pseudo-Frobenius (resp. Frobenius), then so is  $\Omega$ .

**3** If  $\Lambda$  is pseudo-quasi-Frobenius (resp. quasi-Frobenius), then so is  $\Omega$ .

*Proof.* **1** is Lemma 3.17. To prove **2** and **3**, we may assume that  $(R, \mathfrak{m})$  is complete regular local and  $\Lambda$  is a faithful module.

**2.**

$$(K_\Omega)_\Omega \cong \mathrm{Hom}_R(\Lambda \otimes_R RG, R) \cong \mathrm{Hom}_R(\Lambda, R) \otimes (RG)^* \cong K_\Lambda \otimes RG$$

as right  $\Omega$ -modules. It is isomorphic to  $\Lambda_\Omega \otimes RG \cong \Omega_\Omega$  by Lemma 3.20, **1**, since  $K_\Lambda \cong \Lambda$  in  $\mathrm{mod} \Lambda$ . Hence  $\Omega$  is pseudo-Frobenius. If, in addition,  $\Lambda$  is Cohen–Macaulay, then  $\Omega$  is also Cohen–Macaulay, and hence  $\Omega$  is Frobenius.

**3** is proved similarly, using Lemma 3.20, **2**.  $\square$

Note that the assertions for Frobenius and quasi-Frobenius properties also follow easily from Lemma 3.11 and [7, (3.2)].

#### 4. Codimension-two argument

**(4.1)** This section is the second part of this paper. In this section, we show that the codimension-two argument using the existence of 2-canonical modules in [4] is still valid in non-commutative settings, as announced in (1.3).

**(4.2)** Let  $X$  be a locally Noetherian scheme,  $U$  its open subscheme, and  $\Lambda$  a coherent  $\mathcal{O}_X$ -algebra. Let  $i : U \hookrightarrow X$  be the inclusion.

**(4.3)** Let  $\mathcal{M} \in \mathrm{mod} \Lambda$ . That is,  $\mathcal{M}$  is a coherent right  $\Lambda$ -module. Then by restriction,  $i^* \mathcal{M} \in \mathrm{mod} i^* \Lambda$ .

**(4.4)** For a quasi-coherent  $i^* \Lambda$ -module  $\mathcal{N}$ , we have an action

$$i_* \mathcal{N} \otimes_{\mathcal{O}_X} \Lambda \xrightarrow{1 \otimes u} i_* \mathcal{N} \otimes_{\mathcal{O}_X} i_* i^* \Lambda \rightarrow i_*(\mathcal{N} \otimes_{\mathcal{O}_U} i^* \Lambda) \xrightarrow{a} i_* \mathcal{N},$$

where  $u$  is the unit map for the adjoint pair  $(i^*, i_*)$ . So we get a functor  $i_* : \mathrm{Mod} i^* \Lambda \rightarrow \mathrm{Mod} \Lambda$ , where  $\mathrm{Mod} i^* \Lambda$  (resp.  $\mathrm{Mod} \Lambda$ ) denote the category of quasi-coherent  $i^* \Lambda$ -modules (resp.  $\Lambda$ -modules).

**Lemma 4.5.** *Let the notation be as above. Assume that  $U$  is large in  $X$  (that is,  $\mathrm{codim}_X(X \setminus U) \geq 2$ ). If  $\mathcal{M} \in (S'_2)^{\Lambda^{\mathrm{op}}, X}$ , then the canonical map  $u : \mathcal{M} \rightarrow i_* i^* \mathcal{M}$  is an isomorphism.*

*Proof.* Follows immediately from [4, (7.31)].  $\square$

**Proposition 4.6.** *Let the notation be as above, and let  $U$  be large in  $X$ . Assume that there is a 2-canonical right  $\Lambda$ -module. Then we have the following.*

- 1 *If  $\mathcal{N} \in (S'_2)^{i^* \Lambda^{\text{op}}, U}$ , then  $i_* \mathcal{N} \in (S'_2)^{\Lambda^{\text{op}}, X}$ .*
- 2  *$i^* : (S'_2)^{\Lambda^{\text{op}}, X} \rightarrow (S'_2)^{i^* \Lambda^{\text{op}}, U}$  and  $i_* : (S'_2)^{i^* \Lambda^{\text{op}}, U} \rightarrow (S'_2)^{\Lambda^{\text{op}}, X}$  are quasi-inverse each other.*

*Proof.* The question is local, and we may assume that  $X$  is affine.

1. There is a coherent subsheaf  $\mathcal{Q}$  of  $i_* \mathcal{N}$  such that  $i^* \mathcal{Q} = i^* i_* \mathcal{N} = \mathcal{N}$  by [2, Exercise II.5.15]. Let  $\mathcal{V}$  be the  $\Lambda$ -submodule of  $i_* \mathcal{N}$  generated by  $\mathcal{Q}$ . That is, the image of the composite

$$\mathcal{Q} \otimes_{\mathcal{O}_X} \Lambda \rightarrow i_* \mathcal{N} \otimes_{\mathcal{O}_X} \Lambda \rightarrow i_* \mathcal{N}.$$

Note that  $\mathcal{V}$  is coherent, and  $i^* \mathcal{Q} \subset i^* \mathcal{V} \subset i^* i_* \mathcal{N} = i^* \mathcal{Q} = \mathcal{N}$ .

Let  $\mathcal{C}$  be a 2-canonical right  $\Lambda$ -module. Let  $?^\dagger := \underline{\text{Hom}}_{\Lambda^{\text{op}}} (?, \mathcal{C})$ ,  $\Gamma = \underline{\text{End}}_\Lambda \mathcal{C}$ , and  $?^\ddagger := \underline{\text{Hom}}_\Gamma (?, \mathcal{C})$ . Let  $\mathcal{M}$  be the double dual  $\mathcal{V}^{\dagger\ddagger}$ . Then  $\mathcal{M} \in (S'_2)^{\Lambda^{\text{op}}, X}$ , and hence

$$\mathcal{M} \cong i_* i^* \mathcal{M} \cong i_* i^* (\mathcal{V}^{\dagger\ddagger}) \cong i_* (i^* \mathcal{V})^{\dagger\ddagger} \cong i_* (\mathcal{N}^{\dagger\ddagger}) \cong i_* \mathcal{N}.$$

So  $i_* \mathcal{N} \cong \mathcal{M}$  lies in  $(S'_2)^{\Lambda^{\text{op}}, X}$ .

2 follows from 1 and Lemma 4.5 immediately.  $\square$

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